The Relationship Between Immiscible and Miscible Displacement in Porous Media

Y. C. Yortsos
Petroleum and Chemical Engineering
University of Southern California
Los Angeles, CA 90089-1211

Displacement processes in porous media share several common characteristics. Fundamental among them is the representation of the momentum balance via Darcy's law and its multiphase extension (Bear, 1972). Such description leads to systems of equations the solution of which parallels the behavior of the solution of processes in multicomponent chromatography. For instance, it has been recognized that noncapillary, multiphase displacement in one-dimensional (rectilinear or radial) porous media can be solved by techniques identical to those used in chromatographic transport (Helfferich, 1981; Rhee et al., 1986, and references therein). This similarity arises naturally for the case of one-dimensional flow geometries and in the absence of dissipative terms (diffusion, dispersion, or capillarity), where both multiphase and multicomponent chromatography processes are formulated by systems of first-order hyperbolic equations.

In this note we pursue further this relationship in mathematical representation between multiphase flow and chromatographic processes in porous media. Specifically, we consider two-phase, immiscible displacement and single-phase, miscible displacement in the presence of equilibrium adsorption. It is demonstrated that the two problems are mathematically equivalent, regardless of the dimensionality of the flow system and the presence of dissipative effects. Such an analogy is of practical importance in the analysis of various process characteristics, for instance in describing the evolution of unstable two-dimensional disturbances during viscous fingering. The latter is a topic of active current investigations (Homsy, 1987, and references therein), and of fundamental importance in process performance. Any similarities between seemingly different processes would be of considerable help in an effort to reduce complexity and to uncover common mechanisms. An extension to multiphase immiscible displacement and multicomponent miscible displacement is also briefly discussed.

Mathematical Description

We first consider the mathematical description of two-phase (e.g., oil and water) immiscible displacement. Water is assumed to be the displacing and wetting phase. In the dimensionless notation of Yortsos and Hickernell (1987) the following formulation is obtained

$$S_t + \nabla \cdot \underline{u}_w = 0 \tag{1}$$

$$\underline{u}_w = -M\lambda_w \nabla p_w; \quad \underline{u}_o = -\lambda_o \nabla p_o \tag{2}$$

$$\underline{q} = \underline{u}_w + \underline{u}_o \tag{3}$$

$$\nabla \cdot \underline{q} = 0 \tag{4}$$

$$p_o - p_w = P_c \tag{5}$$

In the above S is a normalized water saturation; $\underline{u}_i$, $\lambda_i(S)$, and p_i denote dimensionless quantities for the flow velocity, the normalized mobility, and the pressure of phase i ($i = w, o$), respectively; M is the mobility ratio; $\underline{q}$ the dimensionless flow velocity; and $P_c(S)$ is the dimensionless capillary pressure function. Implicit in the above is the representation of momentum balance of each fluid by the extended Darcy's law and the assumption of capillary equilibrium between the phases.

By suitable rearrangement the system of Eqs. 1–5 can be also cast in the form

$$S_t + \nabla \cdot (\lambda_o \nabla p_o) = 0 \tag{6}$$

$$\underline{q} = -\lambda_T \nabla p_o + \lambda_w \lambda_c \nabla S \tag{7}$$

$$\nabla \cdot \underline{q} = 0 \tag{8}$$

where $\lambda_T(S)$ is the dimensionless total mobility, $\lambda_T = M\lambda_w + \lambda_o$, and $\lambda_c(S)$ denotes a dimensionless capillary pressure gradient ($\lambda_c \approx dP_c/dS < 0$). The system of Eqs. 6–8 is in the standard form frequently used for immiscible displacement calculations.

To proceed further we reformulate Eqs. 6–8 as follows

$$\frac{\partial S}{\partial t} + \underline{q} \cdot \frac{df_w}{dS} \nabla S = \nabla \cdot \left(- \frac{\lambda_o \lambda_w \lambda_c}{\lambda_T} \nabla S \right) \tag{9}$$

$$\underset{\sim}{q} = -\lambda_T \nabla p \tag{10}$$

$$\nabla \cdot \underset{\sim}{q} = 0 \tag{11}$$

where we have introduced

$$f_w = \frac{M\lambda_w}{\lambda_T} \tag{12}$$

$$p \equiv p_o - h(S);\ dh/dS \equiv \lambda_w\lambda_c/\lambda_T \tag{13}$$

The function $f_w(S)$ denotes the fractional flow function (Bear, 1972), here expressed in terms of the normalized variable S. A typical representation of f_w is shown in Figure 1a for various flow conditions. The mathematical variable p physically denotes an effective pressure, while it should be noted that the derivative $h'(S)$ is well defined and nonsingular for systems that are not strongly wet (de Gennes, 1983; Yortsos and Hickernell, 1987).

We next proceed with the formulation of miscible displacement in the presence of equilibrium adsorption, for example, the injection of a dilute polymer solution in a porous medium. In appropriate dimensionless notation the following system is obtained

$$[1 + g(c)]c_t + \underset{\sim}{q}_m \cdot \nabla c = \nabla \cdot (D\nabla c) \tag{14}$$

$$\underset{\sim}{q}_m = -\lambda_m \nabla P \tag{15}$$

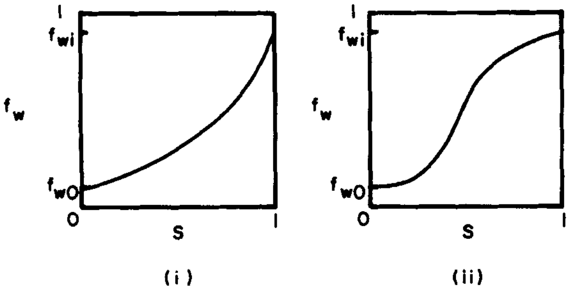
$$\nabla \cdot \underset{\sim}{q}_m = 0 \tag{16}$$

In the above, c denotes a dimensionless concentration; D is a dimensionless dispersion function; $g(c) \equiv dn/dc$ represents the slope of the dimensionless adsorption isotherm for the adsorbed species n under conditions of equilibrium adsorption; $\underset{\sim}{q}_m$ is the flow velocity; $\lambda_m(c)$ is the fluid mobility; and P is the fluid pressure. A typical representation of $n(c)$ is shown in Figure 1b for Langmuir adsorption. Implicit in Eqs. 14–16 is the assumption of dilute solutions, while use is made of the notation of Hickernell and Yortsos (1986) and the formulation by Rhee et al. (1986).

By a proper rearrangement, Eqs. 14–16 can also be cast in the form

$$\frac{\partial m}{\partial t} + \underset{\sim}{q}_m \cdot \frac{1}{1 + g(c)} \nabla m = \nabla \cdot \left[\frac{D}{1 + g(c)} \nabla m\right] \tag{17}$$

$$\underset{\sim}{q}_m = -\lambda_m \nabla P \tag{18}$$



(i) (ii)

Figure 1a. Fractional flow curves.

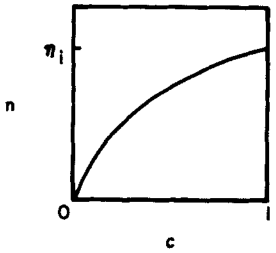


Figure 1b. Adsorption isotherm.

$$\nabla \cdot \underset{\sim}{q}_m = 0 \tag{19}$$

where the variable

$$m \equiv c + n(c) \tag{20}$$

represents a dimensionless total concentration and is an implicit function of c, through the equilibrium adsorption isotherm.

Mathematical Equivalence

We now claim that the two problems described by the systems of Eqs. 9–11 and Eqs. 17–19, respectively, are mathematically equivalent under the following transformation:

$$p = P + \text{const.} \tag{21}$$

$$-\frac{\lambda_o\lambda_w\lambda_c}{\lambda_T} = \frac{D}{1 + g(c)} \tag{22}$$

$$CS + B = m \tag{23}$$

$$A\,\lambda_T(S) = \lambda_m(m) \tag{24}$$

$$f_w(S) = cA/C + E \tag{25}$$

where the constants A, B, C, E are as yet arbitrary. The proof follows by direct substitution. The key to the mathematical equivalence is the introduction of the variable p in Eq. 13, which renders Eqs. 11 and 18 mathematically equivalent.

The above transformation serves to map a two-phase, immiscible displacement into a mathematically equivalent single-phase, miscible displacement in the presence of equilibrium adsorption, and vice versa. Specifically, Eqs. 22 and 23 define the correspondence between capillary and dispersion terms, Eqs. 23 and 25 define the corresponding equilibrium adsorption isotherm from the fractional flow function, while Eqs. 23 and 24 define the total mobility in the miscible displacement problem from that of the immiscible problem. Parameter A in Eq. 24 determines the scaling of the flow velocity between the two problems. It is worth noting that all the information on the two-phase flow characteristics is mapped via Eqs. 23 and 25 into that of the adsorption isotherm, and vice versa. For typical fractional flow and adsorption isotherms the map is one-to-one.

The above transformation preserves specified flux conditions. For instance, in the immiscible case a boundary condition with specified fractional flow value f_w^* reads

$$-(f_w - f_w^*)\,\lambda_T \frac{\partial p}{\partial s} + \frac{\lambda_o\lambda_w\lambda_c}{\lambda_T} \cdot \frac{\partial S}{\partial s} = 0 \tag{26}$$

while in the miscible case a boundary condition with specified concentration c^* reads

$$-D\frac{\partial c}{\partial s} - (c - c^*)\lambda_m \frac{\partial P}{\partial s} = 0 \qquad (27)$$

where s is the direction normal to the boundary surface. Equations 26 and 27 map into each other under Eqs. 21–25, as can be directly shown.

Discussion

The arbitrary constants in Eqs. 21–25 can be fixed by imposing additional conditions so that the corresponding problems are physically meaningful. Specifically, we may require that the end-point values $S = 0, 1$ map into the values $c = 0, 1$, respectively, and that $n(0) = 0$, implying that the dimensional concentration in the miscible problem vanishes when $c = 0$. Then, $B = 0$; $C = 1 + n_i$, where n_i is the adsorbed concentration at $c = 1$; $E = f_w(0) \equiv f_{w0}$; $A/C = f_{wi} - f_{w0}$; $f_w(1) \equiv f_{wi}$. One then obtains from Eqs. 23 and 25 the correspondence

$$m = S(1 + n_i) \qquad (28)$$

$$f_w(m) - f_{w0} = vc \qquad (29)$$

where $v = f_{wi} - f_{w0}$. The parameters $n_i, A > 0$ may be left arbitrary to scale flow rates or other characteristics of the two problems.

Equation 25 (or Eq. 29) suggests that, contrary to intuition, it is the fractional flow curve and not the saturation that maps into the concentration c. On the other hand, Eq. 22 clearly demonstrates the dispersionlike characteristics of the capillary action, as intuitively anticipated. It is also worth noting that in the limit of zero (or constant) adsorption, the corresponding (through Eqs. 21–25) immiscible problem is characterized by a straight-line fractional flow curve. In this limit the mathematical equivalence leads to a physically expected result: straight-line fractional flow curves usually signify that the immiscible problem approaches miscibility conditions.

Finally, it should be added that the above equivalence may become questionable when the issues of anisotropy and flow rate dependence on the dispersion coefficient D or hysteresis on the functions λ_w, λ_o, and λ_c are taken into consideration. Although the subject of rate or anisotropy-dependent relative permeabilities remains open, it is unlikely that the functional relations, if any, would be consistent with those of the miscible displacement problem. At the present stage, the correspondence Eqs. 21–25 should be viewed simply as a convenient mathematical tool that under the conditions stated allows for a unified treatment of the two processes.

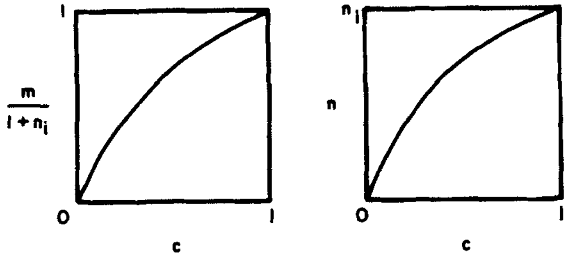


Figure 2a. Adsorption isotherm for fractional flow curve of Figure 1a(i).

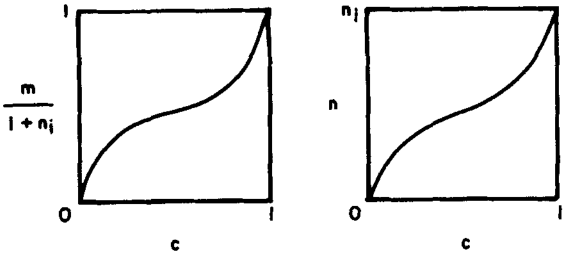


Figure 2b. Adsorption isotherm for fractional flow curve of Figure 1a(ii).

Typical representations of the functional dependence of m and n on c are shown in Figure 2 for the two different fractional flow curves of Figure 1a. Here use is made of Eqs. 28 and 29. It is remarked that the m isotherms are obtained simply by a reflection of the normalized fractional flow curves about the diagonal. Thus, the concave curve in Figure 1a (i) maps into a Langmuir-like convex isotherm in Figure 2a, while the inflection point of Figure 1a (ii) is preserved in the isotherm of Figure 2b. A direct application of the above transformation was successfully utilized by Yortsos and Hickernell (1987) to infer the long-wave stability characteristics of a miscible displacement problem from the corresponding characteristics of the immiscible displacement.

The above analysis may be extended to the case of multiphase displacement and multicomponent adsorption. Here, saturations, fractional flow functions, and concentrations become vectors of dimension N. In the presence of dissipative effects, the equivalence remains valid provided that certain compatibility conditions on the functional dependence of mobility and capillary functions hold. On the other hand, one may directly infer from the hyperbolic part of the equations the equivalence

$$m \equiv c + n = C \cdot S + B \qquad (30)$$

$$f = AC^{-1} \cdot c + E \qquad (31)$$

where m, c, S, B, E are vectors of dimension N, and C is a nonsingular matrix of dimensions $N \times N$.

Acknowledgment

Useful discussions with G. M. Homsy and the hospitality of the Chemical Engineering Department at Stanford University, where part of this work was performed, are gratefully acknowledged.

The author has recently learned that a transformation similar to Eq. 13 was also proposed by G. Chavent, in *Lecture Notes in Mathematics*, A. Dold and B. Eckmann, Eds., **503**, 258 (1976).

Notation

c = concentration
D = dispersion coefficient
f = fractional flow
g = rate of adsorption
m = total concentration
M = mobility ratio
n = adsorbed species concentration
p = pressure
P = pressure
q = flow velocity
S = saturation

t = time
u = phase velocity
λ = mobility

Subscripts

c = capillarity
i = injected
m = miscible
o = oil
T = total
w = water
0 = initial

Literature Cited

Bear, J., *Dynamics of Fluids in Porous Media,* American Elsevier, New York (1972).

de Gennes, P. G., "Theory of Slow Biphasic Flows in Porous Media," *Physicochem. Hydrodynam.*, **4**(2), 175 (1983).

Helfferich, F., "Theory of Multicomponent, Multiphase Displacement in Porous Media," *Soc. Pet. Eng. J.*, 51 (Feb., 1981).

Hickernell, F. J., and Y. C. Yortsos, "Linear Stability of Miscible Displacement Processes in Porous Media in the Absence of Dispersion," *Stud. Appl. Math.*, **74**, 93 (1986).

Homsy, G. M., "Viscous Fingering in Porous Media," *Ann. Rev. Fluid Mech.*, **19**, 271 (1987).

Rhee, H-K, R. Aris, and N. R. Amundson, *Partial Differential Equations 1. Theory of Single First-order Equation,* Prentice-Hall, Englewood Cliffs, NJ (1986).

Yortsos, Y. C., and F. J. Hickernell, "Linear Stability of Immiscible Displacement in Porous Media," submitted SIAM J. in Applied Math. (1987).

Manuscript received Dec. 11, 1986, and revision received April 20, 1987.